A TQFT of Intersection Numbers on Moduli Spaces of Admissible Covers

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Abstract

We construct a two-level weighted TQFT whose structure coefficents are equivariant intersection numbers on moduli spaces of admissible covers. Such a structure is parallel (and strictly related) to the local Gromov-Witten theory of curves in [BP04]. We compute explicitly the theory using techniques of localization on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$. The Frobenius Algebras we obtain are one parameter deformations of the class algebra of the symmetric group S_d. In certain special cases we are able to produce explicit closed formulas for such deformations in terms of the representation theory of S_d.

Introduction

This paper studies a large class of (equivariant) intersection numbers on moduli spaces of admissible covers. For a smooth algebraic curve X, ramified covers of a given degree of X by smooth curves of a given genus are parametrized by moduli spaces called Hurwitz schemes. A smooth compactification of a Hurwitz scheme can be obtained by allowing suitable degenerations, called admissible covers.

Moduli spaces of admissible covers were introduced originally by Harris and Mumford in [HM82]. Intersection theory on these spaces was for a long time extremely hard and mysterious, mostly because they are in general not normal, even if the normalization is always smooth. Only recently in [ACV01], Abramovich, Corti and Vistoli exhibit this normalization as the stack of balanced stable maps of degree 0 from twisted curves to the classifying stack $\mathcal{B}S_d$. This way they attain both the smoothness of the stack and a nice moduli-theoretic interpretation of it. We abuse notation and refer to the Abramovich-Corti-Vistoli (ACV) spaces as admissible covers.

At about the same time, Ionel developed a parallel theory in the symplectic category ([Ion02]) and used push-pull techniques on admissible covers to produce new relations in the tautological ring of $M_{g,n}$ (see also [Ion05]).

In [GV03b], admissible cover loci within the boundary of moduli spaces of stable maps play a key role in establishing the result that the degree $3g-3$ part

of the tautological ring of $\overline{M}_g$ has dimension 1, providing further evidence for a conjecture by Faber, stating that $R(\overline{M}_g)$ is a Gorenstein algebra with socle in degree $3g - 3$.

Recently, in [BGP05], Bryan, Graber and Pandharipande show that the orbifold Gromov-Witten potential of a Gorenstein orbifold can be computed in terms of intersection theory on moduli spaces of admissible covers. With a subtle use of WDVV techniques, they are able to explicitly compute the Gromov-Witten potential for the orbifold $[\mathbb{C}^2/\mathbb{Z}_3]$. Such a computation provides evidence for the crepant resolution conjecture ([BG05]).

We give a few basic definitions and a working description of moduli spaces of admissible covers in section 1.

For all choices of:

- an r-pointed curve $(X, p_1, \ldots, p_n)$;

- a rank two vector bundle $N = L_1 \oplus L_2$ on X, endowed with a natural $\mathbb{C}^* \times \mathbb{C}^*$ action (page 13);

- a vector of partitions $\underline{\eta} = (\eta_1, \ldots, \eta_n)$,

we describe the invariants

$$A_d^h(N) := \int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 p_1, \ldots, \eta_r p_r)}} e^{eq}(-R^\bullet \pi_* f^*(L_1 \oplus L_2)).$$

Motivation for studying these invariants is twofold: they are natural and interesting intersection numbers on their own, that give rise to a beautiful structure. In the context of Gromov-Witten theory, invariants of this form are known as "local" invariants: roughly speaking, they represent the contribution to the Gromov-Witten invariants of a threefold given by rigid curves.

Theorem 1 (page 15) **:** *The invariants $A_d^h(N)$ can be organized to be the structure coefficients of a* $2-$***level, semisimple, weighted TQFT*** *$\mathcal{U}$.*

Section 2 is dedicated to presenting these structures to the unfamiliar reader, while in section 3 the specific TQFT $\mathcal{U}$ is constructed.

The generators for the TQFT are explicitly computed in section 4. The techniques involved are basic dimension counts, reduction to classical intersection theory on moduli spaces of curves, and Atiyah-Bott localization on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$.

An interesting feature of this theory is that the degree 0 part is constructed from Hurwitz numbers. The embedded (see page 11) Frobenius Algebras induced on the Hilbert space by $\mathcal{U}$ are one parameter deformations of the class algebra of the symmetric group, whose TQFT-theoretic description in terms of Hurwitz numbers was studied in the '90s in [DW90] and [FQ93]. An explicit description of such deformations is in general quite complicated. By specializing

to the anti-diagonal action of $\mathbb{C}^*$ inside $\mathbb{C}^* \times \mathbb{C}^*$, it is possible to diagonalize the theory: closed formulas for our invariants and for the deformation are described in terms of the representation theory of the symmetric group S_d (**Theorem 4**, page 30). This is carried out in section 5.

This work is closely connected to and follows recent work of Jim Bryan and Rahul Pandharipande ([BP05],[BP04]), describing the local Gromov-Witten theory of curves.

There, analogous intersection numbers on moduli spaces of (relative) stable maps are organized in a TQFT, that we denote $\mathcal{BP}$. **Theorem** 2 (page 21) shows that the two theories coincide in level $(0,0)$. In all other levels, $\mathcal{U}$ is a normalization of $\mathcal{BP}$ via appropriate powers of a universal generating function factor, which should be understood as the contribution of maps containing contracting components to the Gromov-Witten invariants.

This fact, the most technical result in this paper, is established by computing the genus 0, one-pointed invariants via localization, together with the use of some beautiful Hodge integral computations by Faber-Pandharipande ([FP00]) and Ekedahl-Lando-Shapiro-Vainshtein ([ELSV01] and [GV03a]). The explicit result is:

Theorem 3 (page 23) :*the coefficients for the one-pointed invariants of $\mathcal{U}$ in level $(0,-1)$ are given by the following generating functions:*

$$A_d(0|0,-1)_\eta = (-1)^{d-\ell(\eta)} \frac{\left(2\sin\left(\frac{u}{2}\right)\right)^d}{(s_1)^{\ell(\eta)} \mathfrak{z}(\eta) \prod 2\sin\left(\frac{\eta_i u}{2}\right)}.$$

A direct check in the one-pointed case, together with the semisimplicity of both theories, yield the following corollary:

Corollary 1: *the coefficients of the theories $\mathcal{U}$ and $\mathcal{BP}$ are related by:*

$$A_d(g \mid k_1, k_2)_{\underline{\eta}} = (d!)^{k_1+k_2} s_1^{dk_2} s_2^{dk_1} \mathcal{BP}_d(g, \mid k_1, k_2)_{\underline{\eta}} \mathcal{BP}_d(0 \mid 0, -1)_{(1,\ldots,1)}^{k_1+k_2}$$

This close proximity to Gromov-Witten theory reinforces our interest in moduli spaces of admissible covers, as it anticipates the possibility of a fertile exchange of information between the two contexts. In particular, embedded in the theory $\mathcal{U}^\circ$ (the circle superscript indicates we are restricting our attention to connected covers) we rediscover the classical result:

Aspinwall-Morrison formula:

$$\int_{[\overline{M}_{0,0}(\mathbb{P}^1,d)]} R^1\pi_* f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) = \left(\frac{A_d^{\circ,0}(0 \mid -1,-1)}{u^{2d-2}}\right)_{|u=0} = \frac{1}{d^3}$$

The technique of Atyah-Bott localization suits very well the spaces of admissible covers of a parametrized $\mathbb{P}^1$; the fact that these spaces are smooth

(as DM stacks) requires no need for a virtual fundamental class in order to do intersection theory on them. The modularity of the boundary-fixed loci naturally produces topological recursions that live completely within the realm of admissible covers.

Acknowledgements

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1 Admissible Covers

Moduli spaces of admissible covers are a "natural" compactification of the Hurwitz schemes, parametrizing ramified covers of smooth Riemann Surfaces. The fundamental idea is that, in order to understand limit covers, we allow the base curve to degenerate together with the cover. Branch points are not allowed to "come together"; as two or more branch points tend to collide, a new component of the base curve sprouts from the point of collision, and the points transfer onto it. Similarly, upstairs the cover splits into a nodal cover.

More formally: let $(X, p_1, \cdots, p_r)$ be an r-pointed nodal curve of genus g.

Definition 1. *An **admissible cover** $\pi : E \longrightarrow X$ of degree d is a finite morphism satisfying the following:*

1. *E is a nodal curve.*

2. *Every node of E maps to a node of X.*

3. *The restriction of $\pi : E \longrightarrow X$ to $X \setminus (p_1, \cdots, p_r)$ is étale of constant degree d.*

4. *Nodes can be smoothed. This means: given an admissible cover $\pi : E \to X$, and a node of E, we can find a family of admissible covers $\pi' : E' \to X'$ such that:*

   - *$\pi : E \to X$ is the central fiber of the family;*
   - *locally in analytic coordinates, X', E' and π' are described as follows, for some positive integer n not larger than d:*

$$
\begin{aligned}
E : &\quad e_1 e_2 = a, \\
X : &\quad x_1 x_2 = a^n, \\
\pi : &\quad x_1 = e_1^n, \ x_2 = e_2^n.
\end{aligned}
$$

We recall here the notation we use in this paper, and refer the reader to [Cav05] for a more extensive discussion.

Let $(X, p_1, \ldots, p_r)$ be as before, and $\eta_1, \ldots, \eta_r$ be partitions[1] of the fixed integer d. We denote by

$$\overline{Adm}_{h \overset{d}{\to} X, (\mu_1 p_1, \cdots, \mu_r p_r)}$$

the stack of **possibly disconnected**, degree d admissible covers of the curve X by curves of genus h, such that:

- the ramification profile over the base point p_i is described by the partition η_i;

- all other ramification is required to be simple and is not marked.

The following variations are also used:

connected admissible covers: we add the superscript "$\circ$" to restrict our attention to admissible covers by connected curves;

admissible covers of a genus g curve: we denote by

$$\overline{Adm}_{h \overset{d}{\to} g}$$

the stack of admissible covers of a curve of genus g. This means that also the base curve is allowed to vary in families.

admissible covers of a parametrized $\mathbb{P}^1$: when we intend to fix a parametrization on the base $\mathbb{P}^1$, we write

$$\overline{Adm}_{h \overset{d}{\to} \mathbb{P}^1}.$$

Moduli spaces of admissible covers admit forgetful maps to (quotients of) the configuration spaces of points on the base curve (resp. $\overline{M}_{g,n}$ for admissible covers of a genus g curve)- recording the information of the branch points that are free to move. Tautological ψ classes on admissible covers are defined by pulling back the ψ classes downstairs via these maps.

There is also a natural map from a moduli space of admissible covers of genus h to the corresponding moduli space of curve $\overline{M}_h$, obtained by forgetting the cover map and only remembering the source curve. Tautological λ classes on admissible covers are defined by pulling back λ classes (the Chern classes of the Hodge bundle on the moduli space of stable curves) via these maps.

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[1]See the appendix for partition notation.

1.1 Admissible Covers of a Nodal Curve

Admissible covers of a nodal curve can be described combinatorially in terms of admissible covers of the irreducible components of the curve. This is extremely useful because it opens the way to the use of degeneration techniques and induction. Crucial to this work are the following identities ([Li02]), that take place in the Chow ring with rational coefficients.

Reducible nodal curve: let

$$X = X_1 \bigcup_{x_1 = x_2} X_2$$

be a nodal curve of genus g, obtained by attaching at a point two irreducible curves of genus g_1 and g_2. Then:

$$[\overline{Adm}_{h \xrightarrow{d} X}] = \sum_{\eta, h_1, h_2} \mathfrak{z}(\eta) [\overline{Adm}_{h_1 \xrightarrow{d} X_1, (\eta)}] \times [\overline{Adm}_{h_2 \xrightarrow{d} X_2, (\eta)}], \quad (1)$$

where:

- $\eta = ((\eta^1)^{m_1}, \dots, (\eta^k)^{m_k})$ runs over all partitions of d;
- we define the combinatorial factor:

$$\mathfrak{z}(\eta) := \prod m_i! (\eta^i)^{m_i}. \quad (2)$$

  In particular, $\mathfrak{z}(\eta)$ is the order of the centralizer in S_d of any group element in the conjugacy class of η;
- $h_1 + h_2 + \ell(\eta) - 1 = h$.

Note: if we are dealing with admissible cover spaces with also a prescribed vector of ramification conditions $\underline{\mu}$, analogous formulas hold; the μ_i's need to be distributed on the two twigs X_1 and X_2 in all possible ways.

Irreducible nodal curve: let

$$X = X'/\{x_1 = x_2\}$$

be a nodal curve of genus g, obtained by gluing two distinct points of an irreducible curve X' of genus $g - 1$. As an element in the Chow ring with rational coefficients, we can then express:

$$[\overline{Adm}_{h \xrightarrow{d} X}] = \sum_{\eta} \mathfrak{z}(\eta) [\overline{Adm}_{h' \xrightarrow{d} X', (\eta, \eta)}], \quad (3)$$

where the sum is over all partitions η of d, and h' is determined by the relation:

$$h' + \ell(\eta) = h.$$

2 Topological Quantum Field Theories

An excellent and elementary reference for two-dimensional topological quantum field theories in mathematics is [Koc03].

Definition 2. *A (1+1) dimensional topological quantum field theory is a functor of tensor categories:*

$$\mathcal{T} : \boldsymbol{2Cob} \longrightarrow \ \boldsymbol{Free\ Rmod}.$$

On the right hand side, we have the category of free modules over a commutative ring R. Let us describe the category **2Cob**:

objects: objects are one-dimensional oriented closed manifolds, i.e., finite disjoint unions of oriented circles.

morphisms: morphisms are (equivalence classes of) oriented cobordisms between two objects. We can think of them as oriented topological surfaces with oriented boundary components.

composition: we compose two morphisms by simply concatenating them; equivalently, we glue negatively oriented boundary components of one surface to positively oriented boundary components of the other.

tensor structure: the tensor operation is disjoint union.

The free module $H := \mathcal{T}(S^1)$ is called the **Hilbert space** of the TQFT.

All topological surfaces can be decomposed into discs, annuli, and pairs of pants. Therefore, the structure of a TQFT is completely determined if it is described on these basic building blocks.

2.1 Tensor Notation

It is convenient, for explicit computations, to familiarize ourselves with tensor notation for TQFT's. Let us explicitly choose a basis $e_1, \ldots, e_r$ for the Hilbert space H, and let us denote the dual basis by $e^1, \ldots, e^r$. Let $W_m^n(g)$ be a genus g cobordism from m to n circles. Then the map

$$\mathcal{T}(W_m^n(g)) : H^{\otimes m} \to H^{\otimes n}$$

can be thought of as a vector in $(H^*)^{\otimes m} \otimes H^{\otimes n}$. We denote by

$$\Gamma(W_m^n(g))_{i_1,\ldots,i_m}^{j_1,\ldots,j_n}$$

the coefficient of $\mathcal{T}(W_m^n(g))$ in the direction of the basis element $e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_n}$. That is,

$$\mathcal{T}(W_m^n(g)) = \sum \Gamma(W_m^n(g))_{i_1,\ldots,i_m}^{j_1,\ldots,j_n} e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_n},$$
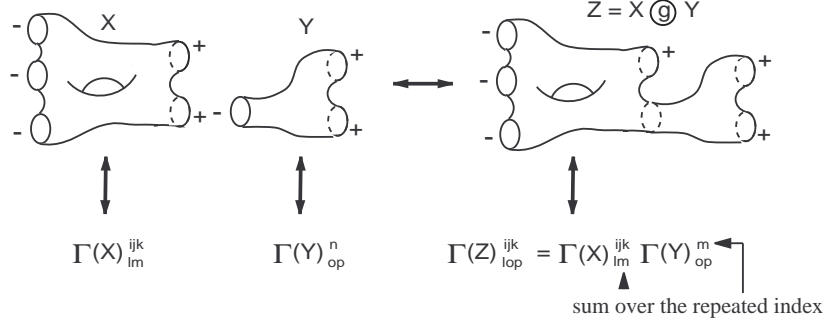
as depicted in Figure 1.

Figure 1: Gluing in tensor notation.

2.2 Frobenius Algebras

A TQFT gives the Hilbert space H the structure of a commutative Frobenius algebra. This means it defines an associative and commutative multiplication "$\cdot$" and an inner product (also called the metric of the TQFT) "$<,>$" on H such that

$$< h_1 \cdot h_2, h_3 >=< h_1, h_2 \cdot h_3 > \tag{4}$$

holds for all h_1,h_2,h_3 in the Hilbert space H. It is easy to see how the structure is induced: multiplication is the map associated to the $(-,-,+)$-pair of pants, the inner product is the scalar map associated to the $(-,-)$-annulus. As a consequence, we see immediately that the cap with positively oriented boundary corresponds to the unit vector for the multiplication map just defined, whereas the $(-)$-cap corresponds to the counit operator in the Frobenius Algebra.

2.3 Semisimple TQFT's

Definition 3. *A TQFT $\mathcal{T}$ is semisimple if the Frobenius algebra induced on the Hilbert space H is semisimple. That is, if there is an orthonormal basis $e_1, \ldots e_r$ for H such that*

$$e_i \cdot e_j = \delta_{ij} e_i.$$

An equivalent point of view is to say that $\mathcal{T}$ is a direct sum

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r,$$

where all $\mathcal{T}_i$'s are TQFT with Hilbert space equal to the ground ring.

Denote by $e_1, \ldots, e_r$ a semisimple basis for H. We can also think of e_i being the identity vector for the space H_i. Let $e^1, \ldots, e^r$ be the dual basis. Then semisimplicity is equivalent to asking all non-diagonal coefficients to vanish:

$$\Gamma^{j_1, \ldots, j_m}_{i_1, \ldots, i_n}(W^n_m(g)) = 0,$$

8

unless $i_1 = i_2 = \cdots = i_n = j_1 = \cdots = j_m$.

There are now r universal constants $\lambda_1, \ldots, \lambda_r$ that govern the structure of the TQFT. They can be defined in many equivalent ways. Here are two equivalent descriptions that we will be using later on:

1. $1/\lambda_i$ is the image of the basis vector e_i via the counit operator.

2. λ_i is the i-th eigenvalue of the genus adding operator (this is the linear map associated to the torus with a negative and a positive puncture, represented in Figure 3).

Now the following structure theorem holds:

Fact 1. *Let $\mathcal{T}$ be a semisimple TQFT, and all notation as above. Denote by $W_m^n(g)$ a genus g surface with m input and n output holes. Then:*

$$\mathcal{T}(W_m^n(g)) = \sum_{i=1}^{r} \lambda_i^{g+n-1} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \ \ times} \otimes \underbrace{e_i \otimes \cdots \otimes e_i}_{n \ \ times}.$$

In particular:

$$\mathcal{T}(W_0^0(g)) = \sum_{i=1}^{r} \lambda_i^{g-1}. \tag{5}$$

2.4 The TQFT of Hurwitz Numbers

In the early 1990s ([DW90]), the Robbert Dijkgraaf noticed that a TQFT approach yields a beautiful and elegant solution to a classical mathematical problem: counting ramified and unramified covers of a topological surface.

Let $(X, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r + s)$-marked smooth topological surface. Let $\underline{\eta} = (\eta_1, \ldots, \eta_r)$ be a vector of partitions of the integer d. We define the *Hurwitz number*:

$$H_d^{h,X}(\underline{\eta}) \quad := \quad \text{weighted number of} \left\{ \begin{array}{c} \textit{degree d covers} \\ C \xrightarrow{\pi} X \textit{ such that}: \\ \bullet\ C \textit{ is a surface of genus h}; \\ \bullet\ \pi \textit{ is unramified over} \\ X \setminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\}; \\ \bullet\ \pi \textit{ ramifies with profile } \eta_i \textit{ over } p_i; \\ \bullet\ \pi \textit{ has simple ramification over } q_i. \end{array} \right\}$$

The above number is weighted by the number of automorphisms of such covers.

For a Hurwitz number to be nonzero, s, h and $\underline{\eta}$ must satisfy the Riemann Hurwitz formula. This is why we omit s from the above notation. In particular, if we require $s = 0$, then (at most one value of) h is determined by $\underline{\eta}$. We denote by $H_d^X(\underline{\eta})$ the corresponding Hurwitz number.

We define the TQFT $\mathcal{D}$ as follows:

1. the ground field is $\mathbb{C}$;

2. the Hilbert space is $H = \bigoplus_{\eta \vdash d} \mathbb{C}e_\eta$;

3. morphisms are assigned according to the prescription:

$$
\begin{array}{cccccc}
 & \xrightarrow{\mathcal{D}} & \mathcal{D}(X) & : & H^{\otimes n} & \longrightarrow & \mathbb{C} \\
 & & & & e_{\eta_1} \otimes \ldots \otimes e_{\eta_n} & \mapsto & H_d^X(\underline{\eta}).
\end{array}
$$

$$
\xrightarrow{\mathcal{D}} \quad \mathcal{D}(A) = \sum \mathfrak{z}(\eta)e_\eta \otimes e_\eta.
$$

Fact 2 (Dijkgraaf, Witten/ Freed, Quinn). *The above assignment defines a semisimple TQFT $\mathcal{D}$. Let η be a partition of d, representing a conjugacy class of the symmetric group, and let h be an element in this conjugacy class. Via the identification:*

$$
e_\eta = \frac{1}{d!} \sum_{g \in S_d} g^{-1}hg,
$$

the Hilbert space is isomorphic, as a Frobenius algebra, to the class algebra of the symmetric group in d letters, $\mathcal{Z}(\mathbb{C}[S_d])$.

A semisimple basis is indexed by irreducible representations ρ of S_d. Let ρ be such a representation and $\mathcal{X}_\rho$ its character function, then:

$$
e_\rho = (dim\rho) \sum_{\eta \vdash d} \mathcal{X}_\rho(\eta) e_\eta.
$$

This allows Dijkgraaf to recover the classical Burnside formula, expressing the number of unramified covers of a genus g curve as:

$$
\boxed{H_d^{gd-d+1,g}(\phi) = \sum_\rho \left(\frac{d!}{\dim\rho} \right)^{2g-2}.} \tag{6}
$$

2.5 Weighted TQFT's

A weighted TQFT contains some extra structure with respect to an ordinary TQFT. Every cobordism comes equipped with a sequence of weights, or levels. When you concatenate two cobordisms, you add the levels componentwise. We are in particular interested in the theory with 2 levels.

Define the category $\mathbf{2Cob}^{k_1,k_2}$ as follows:

1. Objects and tensor structure are the same as in **2Cob**.

2. Morphisms are given by triples (W, k_1, k_2), where W is an oriented cobordism as in **2Cob**, k_1, k_2 are two integers called levels.

3. Composition of morphisms consists in concatenating the cobordisms and adding the levels componentwise.

Definition 4. *A weighted TQFT is a functor of tensor categories:*

$$\mathcal{WT} : \mathbf{2Cob^{k_1,k_2}} \longrightarrow \mathbf{FRMod}.$$

It is immediate that if we restrict our attention to only cobordisms with weight $(0,0)$, we obtain an ordinary TQFT. More generally, there are a $\mathbb{Z} \times \mathbb{Z}$ worth of ordinary TQFTs embedded in a weighted TQFT. Denote by $\mathcal{X}$ the Euler characteristic of a cobordism W. For any $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, restricting the weighted TQFT to cobordisms with level

$$(a\mathcal{X}, b\mathcal{X})$$

yields an ordinary TQFT.

2.6 Generation Results

There are several possible ways to generate a weighted TQFT. A particularly natural one consists in generating the level $(0,0)$ TQFT, and then giving natural operators that allow one to shift the levels. These elements can be chosen to be, for example, the cylinders with weight $(\pm 1, 0)$ and $(0, \pm 1)$. These operators change the levels of the cobordisms without altering its topology. An equivalent, and equally natural choice, is given by the caps, as illustrated in Figure 2.

In particular, it is immediate to see that A (resp. C) is the inverse of B (resp. D) in the level $(0,0)$ Frobenius algebra. Hence the following generation result.

Fact 3 (Bryan-Pandharipande, [BP04] 4.1). *A weighted TQFT $\mathcal{WT}$ is uniquely determined by a commutative Frobenius algebra over k for the level $(0,0)$ theory and by two distinguished invertible elements in the Frobenius algebra:*

$$\mathcal{WT} \left(\begin{array}{c} \bigcirc\hspace{-1em}\big| \; + \\ (-1,0) \end{array} \right), \quad \mathcal{WT} \left(\begin{array}{c} \bigcirc\hspace{-1em}\big| \; + \\ (0,-1) \end{array} \right).$$
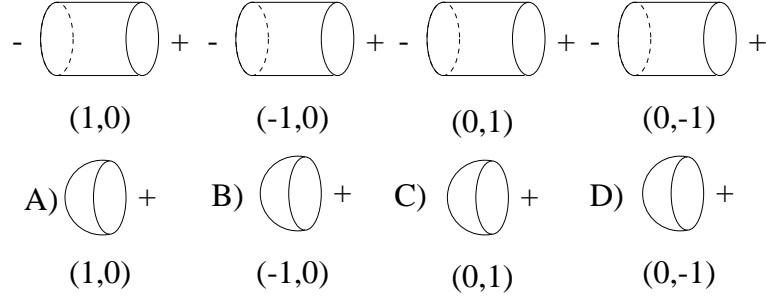
Figure 2: Level changing objects.

2.7 Semisimple Weighted TQFT

A weighted TQFT of rank r is semisimple if there is a basis for the Hilbert space such that all the non-zero tensors in the theory are diagonal. This is equivalent to asking that all embedded ordinary TQFT's are semisimple (possibly with rescaled semisimple bases). Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of the level $(0,0)$ genus adding operator. Let $\mu_1, \ldots, \mu_r$ be the eigenvalues of the level $(-1,0)$ annulus, and $\overline{\mu}_1, \ldots, \overline{\mu}_r$ be the eigenvalues for the level $(0,-1)$ annulus, as illustrated in Figure 3.
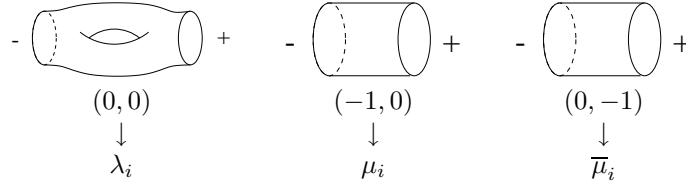

Figure 3: The genus-adding and the level-changing operators.

Fact 4 (Bryan-Pandharpiande,[BP04], 5.2). *Let $\mathcal{WT}$ be a semisimple TQFT. Denote by $W_m^n(g|k_1, k_2)$ a cobordism of genus g between m input and n output holes, of level (k_1, k_2). Then:*

$$\mathcal{T}(W_m^n(g|k_1, k_2)) = \sum_{i=1}^r \lambda_i^{g+n-1} \mu_i^{-k_1} \overline{\mu}_i^{-k_2} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \ \ times} \underbrace{\otimes e_i \otimes \cdots \otimes e_i}_{n \ \ times}.$$

In particular:

$$\mathcal{T}(W_0^0(g|k_1, k_2)) = \sum_{i=1}^r \lambda_i^{g-1} \mu_i^{-k_1} \overline{\mu}_i^{-k_2}. \tag{7}$$

Observation: the following equivalent definitions can be given for the quantities λ_i, μ_i and $\overline{\mu}_i$. Denote by $e_1, \ldots, e_r$ the vectors of a semisimple basis for the weighted TQFT $\mathcal{WT}$:

- λ_i^{-1} is the value of the level $(0,0)$ counit on e_i:

$$\mathcal{WT}\left(\begin{array}{c} - \bigcirc \\ (0,0) \end{array} \right)(e_i) = \lambda_i^{-1}.$$

- μ_i is the coefficient of e_i in the level $(-1,0)$ +disc vector:

$$\mathcal{WT}\left(\begin{array}{c} \bigcirc + \\ (-1,0) \end{array} \right) = \sum \mu_i e_i.$$

- $\overline{\mu}_i$ is the coefficient of e_i in the level $(0,-1)$ +disc vector:

$$\mathcal{WT}\left(\begin{array}{c} \bigcirc + \\ (0,-1) \end{array} \right) = \sum \overline{\mu}_i e_i.$$

3 Construction of the Theory

3.1 The Admissible Covers Invariants

Let $(X, p_1, \ldots, p_r)$ be a smooth, irreducible, projective curve of genus g with r distinct marked points, and $N = L_1 \oplus L_2$ a rank 2 vector bundle on X. The torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts naturally on N: the first coordinate scales (with weight one) the fibre of L_1, the second coordinate scales the fibre of L_2.

The T-equivariant cohomology of a point is a polynomial ring in two indeterminates, that we denote:

$$H_T^*(pt) = \mathbb{C}[s_1, s_2].$$

We are interested in the following class of intersection numbers:

$$A_d^h(N) := \int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 p_1, \ldots, \eta_r p_r)}} e^{eq}(-R^\bullet \pi_* f^*(L_1 \oplus L_2)),$$

where:

- $\overline{Adm}_{h \xrightarrow{d} X(\eta_1 p_1, \ldots, \eta_r p_r)}$ is as defined on page 4.

- e^{eq} is the equivariant Euler class of the virtual bundle in question.

- π is the universal family over the space of admissible covers.

- f is the universal cover map followed by the canonical contraction map to X.

By [BP01], this integral only depends on the genus g of the curve X and on the degrees k_1 and k_2 of the line bundles L_1 and L_2. In our forthcoming TQFT formulation it will be useful to emphasize this fact, so we choose to denote the above invariants:

$$A_d^h(N)_{\underline{\eta}} = A_d^h(g|k_1, k_2)_{\underline{\eta}}.$$

We consider these invariants for all genera h, and organize them in generating function form as follows:

$$A_d(g|k_1, k_2)_{\underline{\eta}} := \sum_{h \in \mathbb{Z}} u^{\star(h)} A_d^h(g|k_1, k_2)_{\underline{\eta}}. \qquad (8)$$

The appropriate exponent for the generating function is defined:

$$\star(h) = dim(\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 p_1, \ldots, \eta_r p_r)}) = 2h - 2 + d(2 - 2g - r) + \sum_{i=1}^{r} \ell(\eta_i).$$

By expanding the equivariant Euler class in terms of ordinary Chern classes and of the equivariant parameters, we can express these invariants in terms of nonequivariant integrals. Let $h \in \mathbb{Z} \cup \phi$ be a function of b_1, b_2 determined by the equation

$$b_1 + b_2 = dim(\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}) = 2h - 2 + d(2 - 2g - r) + \sum_{i=1}^{r} \ell(\eta_i).$$

Define

$$A_d^{b_1, b_2}(g|k_1, k_2)_{\underline{\eta}} := \int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}} c_{b_1}(-R^\bullet \pi_* f^*(L_1)) c_{b_2}(-R^\bullet \pi_* f^*(L_2)).$$

Then the relative invariants are:

$$A_d(g|k_1, k_2)_{\underline{\eta}} := \sum_{b_1 + b_2 = 0}^{\infty} u^{b_1 + b_2} s_1^{r_1 - b_1} s_2^{r_2 - b_2} A_d^{b_1, b_2}(g|k_1, k_2)_{\underline{\eta}}. \qquad (9)$$

This shows that the partition function for our invariants is a Taylor series in u, whose coefficients are rational functions in s_1 and s_2. The degree of these rational functions is independent of h. It is equal to $r_1 + r_2 - b_1 - b_2 = d(2g - 2 - r) - \sum_{i=1}^{r} \ell(\eta_i)$.

3.2 The Weighted TQFT $\mathcal{U}$

We construct a weighted TQFT $\mathcal{U}$, whose structure coefficients encode the invariants just presented.

The ground ring is defined to be $R = \mathbb{C}[[u]](s_1, s_2)$.

The Hilbert space of the theory is a free R-module of rank equal to the number of partitions of the integer d. A privileged basis is indexed by such partitions η.

$$\mathcal{H} = \bigoplus_{\eta \vdash d} R e_\eta.$$

We denote the dual space by $\mathcal{H}^*$, and the dual basis vectors by e^η.

In order to construct our TQFT we reason topologically. We think of the marked points on a curve $(X, x_1, \ldots, x_{r+s})$ as of punctures that we can "enlarge" into loops. We can assign positive or negative orientation to such loops, and arrange the negatively oriented loops $x_1, \ldots, x_r$ to the left, the positively oriented to the right (after relabelling $x_{r+i} = y_i$). We now have an oriented cobordism.

To completely determine the structure of the theory we define the scalar maps associated to arbitrary cobordisms into the empty set, and the coproduct, that allows us to "move" boundary components from the left to the right.

$$\mathcal{U}(X) \quad : \qquad H^{\otimes r} \qquad \longrightarrow \quad \mathbb{C}[[u]](s_1, s_2)$$

$$\mathrm{e}_{\eta_1} \otimes \ldots \otimes \mathrm{e}_{\eta_r} \quad \mapsto \quad A_d(g|k_1, k_2)_{\underline{\eta}}.$$

$$\overset{\mathcal{U}}{\mapsto} \quad \mathcal{U}(A) = \sum_{\eta \vdash d} \mathfrak{z}(\eta)(s_1 s_2)^{\ell(\eta)} \mathrm{e}_\eta \otimes \mathrm{e}_\eta.$$

The combinatorial factor $\mathfrak{z}(\eta)$ is defined in page 6.

Theorem 1. *The structure $\mathcal{U}$ defined in the previous paragraph is a two-level, weighted semisimple TQFT.*

In practical terms, it is often very convenient to adopt the conventional riemannian geometry tensor notation. If $X = (X, x_1, \ldots, x_r, y_1, \ldots, y_s | k_1, k_2)$ represents a cobordism of genus g and level k_1, k_2 from r circles to s circles, then $\mathcal{U}(X)$ is an element of $(\mathcal{H}^*)^{\otimes r} \otimes \mathcal{H}^{\otimes s}$. We denote by

$$A_d(g|k_1, k_2)^{\mu_1, \ldots, \mu_s}_{\eta_1, \ldots, \eta_r}$$

the coordinate of $\mathcal{U}(X)$ in the direction of the basis element $\mathrm{e}^{\eta_1} \otimes \cdots \otimes \mathrm{e}^{\eta_r} \otimes \mathrm{e}_{\mu_1} \otimes \cdots \otimes \mathrm{e}_{\mu_s}$.

With this notation, the coproduct gives the following formula for raising and lowering indices:

$$A_d(g|k_1, k_2)^{\mu_1, \ldots, \mu_s}_{\eta_1, \ldots, \eta_r} = \left(\prod_{i=1}^{s} \mathfrak{z}(\mu_i)(s_1 s_2)^{\ell(\mu_i)} \right) A_d(g|k_1, k_2)_{\eta_1, \ldots, \eta_r, \mu_1, \ldots, \mu_s}. \quad (10)$$

3.3 Proof of Theorem 1

Proving that $\mathcal{U}$ is indeed a weighted TQFT amounts to verifying the following three statements:

identity: the tensor associated to the level $(0,0)$ trivial cobordism from the circle to the circle is the identity morphism of the Hilbert space $\mathcal{H}$.

gluing two curves: for any two vectors $\underline{\eta}, \underline{\mu}$ of partitions of d, and integers satisfying $g = g' + g''$, $k_1 = k_1' + k_1''$, $k_2 = k_2' + k_2''$,

$$A_d(g|k_1, k_2)_{\eta_1,\ldots,\eta_r}^{\mu_1,\ldots,\mu_s} = \sum_{\nu \vdash d} A_d(g'|k_1', k_2')_{\eta_1,\ldots,\eta_r}^{\nu} A_d(g''|k_1'', k_2'')_{\nu}^{\mu_1,\ldots,\mu_s}. \quad (11)$$

self-gluing: for any vector of partitions $\underline{\eta}$, and integers g, k_1, k_2,

$$A_d(g + 1|k_1, k_2)_{\eta_1,\ldots,\eta_r} = \sum_{\nu \vdash d} A_d(g|k_1, k_2)_{\eta_1,\ldots,\eta_r,\nu}^{\nu}. \quad (12)$$

3.3.1 Identity

This fact is easily proven. One very clever way to do it, which is pursued in [BP05], is to notice that the degree 0 coefficients in our TQFT agree with the classical TQFT of Hurwitz numbers constructed by Dijkgraaf in [DW90] and recalled in section 2.4. The vanishing of all higher degree terms can be obtained as a straightforward consequence of the gluing laws, or simply by showing that the dimensions of the moduli spaces in question exceed the maximum degree of a non-equivariant class in the integrand.

3.3.2 Gluing Two Curves

In order to minimize the burden of bookkeeping, we prove the result when $r = s = 0$ (i.e., the resulting glued curve is not marked). In the general case, the proof follows exactly the same steps, and all the extra indices are simply carried along for the ride.

Consider a one parameter family of genus g curves W, and the corresponding map to the moduli space,

$$\begin{array}{c} W \\ \downarrow \\ \varphi : \quad \mathbb{A}^1 \quad \to \quad \overline{M}_g, \end{array}$$

such that:

- the central fiber

$$W_0 = X_1 \bigcup_{b_1 = b_2} X_2$$

  is a nodal curve obtained by attaching at a point two smooth curves of genus g' and g'' (with $g' + g'' = g$);

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- all other fibers $W_s, s \neq 0$, are smooth curves of genus g.

Consider the moduli space $\overline{Adm}_{h \xrightarrow{d} g}$ of admissible covers of a genus g curve by a genus h curve, all ramification simple. By [ACV01], there is a flat morphism

$$\overline{Adm}_{h \xrightarrow{d} g} \to \overline{M}_g,$$

We can construct the following cartesian diagram:

$$\mathcal{A}_s = \overline{Adm}_{h \xrightarrow{d} W_s} \quad \hookrightarrow \quad \mathcal{A} \quad \to \quad \overline{Adm}_{h \xrightarrow{d} g}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\{s\} \qquad\quad \hookrightarrow \quad \mathbb{A}^1 \quad \to \quad \overline{M}_g$$

The stack $\mathcal{A}$ must be thought as of the stack of relative admissible covers of the family W. For $s \neq 0$, we obtain admissible covers of a smooth genus g curve; for $s = 0$, we recover admissible covers of the nodal curve W_0.

It is possible to construct two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W, with the following properties:

1. $\mathcal{L}_i$ restricted to any fiber W_s is a line bundle $L_{i,s}$ of degree k_i.

2. Over the central fiber W_0, $\mathcal{L}_i$ restricts to a line bundle $L'_{i,s}$ of degree k'_i on X_1, and restricts to a line bundle $L''_{i,s}$ of degree k''_i on X_2.

3. $\mathbb{C}^*$ acts naturally on $\mathcal{L}_i$ by scaling the fibers (with weight one).

Consider the following diagram:

$$\mathcal{U}_{\mathcal{A}} \quad \xrightarrow{f} \quad \mathcal{W} \quad \to \quad W$$

$$\pi \downarrow \quad \swarrow$$

$$\mathcal{A}$$

$\mathcal{U}_{\mathcal{A}}$ is the universal family of the moduli space $\mathcal{A}$, $\mathcal{W}$ is the universal target and f the universal admissible cover map.

The pull-push

$$\mathcal{I} = -R^\bullet \pi_* f^* (\mathcal{L}_1 \oplus \mathcal{L}_2)$$

is a virtual bundle of virtual rank $r = 2g - 2 - d(k_1 + k_2)$.

By the flatness of the family $\mathcal{A}$ over $\mathbb{A}^1$, the integral of the top Chern class $c_r(\mathcal{I})$ restricted to a fiber $\mathcal{A}_s$ is independent of the fiber. For $s \neq 0$, we obtain

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_s}} c_r(\mathcal{I}\,|_s) = A_d^h(g|k_1, k_2) \qquad (13)$$

We want to evaluate the same expression restricted to s=0, and show it equals the right hand side of (11). We choose to show the equality at the

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generic genus h degree of the generating function, to emphasize the geometric nature of the construction. We hence need to establish the following claim, which consists of expanding the genus h term in equation (11), and lowering indices as in (10).

Claim 1.

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_0}} c_r(\mathcal{I} \mid_0) = \sum_{\nu \vdash d} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1, h_2} A_d^{h_1}(g' \mid k_1', k_2')_\nu A_d^{h_2}(g'' \mid k_1'', k_2'')_\nu,$$

where the second sum is over pairs of indices such that $h_1 + h_2 + \ell(\nu) - 1 = h$.

PROOF: Recall that, by (1) in section 1.1,

$$[\overline{Adm}_{h \xrightarrow{d} W_0}] = \sum_{\nu \vdash d} \mathfrak{z}(\nu) \sum_{h_1, h_2} [\overline{Adm}_{h_1 \xrightarrow{d} X_1, (\nu b_1)}] \times [\overline{Adm}_{h_2 \xrightarrow{d} X_2, (\nu b_2)}],$$

where:

- $h_1 + h_2 + \ell(\nu) - 1 = h$;

- $dim(\overline{Adm}_{h_1 \xrightarrow{d} X_1, (\nu b_1)}) + dim(\overline{Adm}_{h_2 \xrightarrow{d} X_2, (\nu b_2)}) = dim(\overline{Adm}_{h \xrightarrow{d} W_0})$.

Consider the pull-back of the normalization sequence associated to the restriction of $\mathcal{L}_i$ to W_0:

$$0 \to f^*(L_{i,0}) \to f^*(L_{i,0}') \oplus f^*(L_{i,0}'') \to f^*(L_{i,0}) \mid_{X_1 \cap X_2} \to 0.$$

This sequence yields a long exact sequence of higher direct image sheaves

$$0 \to R^0 \pi_* f^*(L_{i,0}) \to R^0 \pi_* f^*(L_{i,0}') \oplus R^0 \pi_* f^*(L_{i,0}'') \to R^0 \pi_* f^*(L_{i,0}) \mid_{X_1 \cap X_2} \to$$

$$\to R^1 \pi_* f^*(L_{i,0}) \to R^1 \pi_* f^*(L_{i,0}') \oplus R^1 \pi_* f^*(L_{i,0}'') \to 0.$$

Notice that $(L_{i,0}) \mid_{X_1 \cap X_2}$ is a skyscraper sheaf $\mathbb{C}_b$, on which $\mathbb{C}^*$ acts with weight 1.

Let us restrict our attention to a connected component of $\mathcal{A}_0$ on which the covers split as two smooth covers of genus h_1 and h_2, with ramification profile ν over the shadows of the node. Here, $f^*(L_{i,0}) \mid_{X_1 \cap X_2}$ is a trivial vector bundle of rank $\ell(\nu)$, endowed with a natural $\mathbb{C}^*$ action.

From this fact and the exact sequence above we conclude:

$$c_{r_i}(-R^\bullet \pi_* f^*(L_{i,0})) = s_i^{\ell(\nu)} c_{r_i'}(-R^\bullet \pi_* f^*(L_{i,0}')) c_{r_i''}(-R^\bullet \pi_* f^*(L_{i,0}'')),$$

and finally

$$c_r(\mathcal{I} \mid_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathcal{I} \mid_0') c_{r''}(\mathcal{I} \mid_0''). \tag{14}$$

Now clinching the claim is just a matter of putting everything together:

$$\int_{\overline{Adm}_{h\xrightarrow{d}W_0}} c_r(\mathcal{I}\,|_0) \;=\; \sum_\nu \mathfrak{z}(\nu) \sum_{h_1,h_2} \int_{\overline{Adm}_{h_1\xrightarrow{d}X_1,(\nu b_1)}\times\overline{Adm}_{h_2\xrightarrow{d}X_2,(\nu b_2)}} c_r(\mathcal{I}\,|_0)$$

$$=\; \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} \int_{\overline{Adm}_{h_1\xrightarrow{d}X_1,(\nu b_1)}} c_{r'}(\mathcal{I}\,|_0') \int_{\overline{Adm}_{h_2\xrightarrow{d}X_2,(\nu b_2)}} c_{r''}(\mathcal{I}\,|_0'')$$

$$=\; \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} A_d^{h_1}(g'|k_1',k_2')_\nu A_d^{h_2}(g''|k_1'',k_2'')_\nu \quad.$$

3.3.3 Self-gluing

The structure of the proof is very similar to the previous case. Again, we simplify the notation by assuming $r = 0$.

Consider a one parameter family of genus g curves W, and the corresponding map into the moduli space,

$$\varphi:\quad \begin{array}{c} W \\ \downarrow \\ \mathbb{A}^1 \end{array} \quad\to\quad \overline{M}_g,$$

such that:

- the central fiber

  $$W_0 = X/\{b_1 = b_2\}$$

  is a nodal curve obtained by identifying two distinct points on an irreducible smooth curve X of genus $g-1$;

- all other fibers $W_s, s \neq 0$, are smooth curves of genus g.

As before, we can construct the following cartesian diagrams:

$$\begin{array}{ccccc} \mathcal{A}_s = \overline{Adm}_{h\xrightarrow{d}W_s} & \hookrightarrow & \mathcal{A} & \to & \overline{Adm}_{h\xrightarrow{d}g} \\ \downarrow & & \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & \mathbb{A}^1 & \to & \overline{M}_g \end{array}$$

and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W, with the following properties:

1. $\mathcal{L}_i$ restricted to any fiber W_s is a line bundle $L_{i,s}$ of degree k_i.

2. Over the central fiber W_0, $\mathcal{L}_i$ pulls back to a line bundle $L_{i,s}'$ of degree k_i on the normalization X.

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3. $\mathbb{C}^*$ acts naturally on $\mathcal{L}_i$ by scaling the fibers (with weight one).

We now consider the equivariant top Chern class of the pull-push

$$\mathcal{I} = -R^\bullet \pi_* f^*(\mathcal{L}_1 \oplus \mathcal{L}_2).$$

For $s \neq 0$,

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_s}} c_r(\mathcal{I}\mid_s) = A_d^h(g|k_1, k_2). \tag{15}$$

Again, we can show that the corresponding integral over the central fiber yields exactly the genus h expansion of the right hand side of equation (12).

Claim 2.

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_0},} c_r(\mathcal{I}\mid_0) = \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k_1, k_2)_{\nu,\nu} \quad,$$

where $h' + \ell(\nu) = h$.

PROOF: By (3) in section 1.1,

$$[\overline{Adm}_{h \xrightarrow{d} W_0}] = \sum_{\nu \vdash d} \mathfrak{z}(\nu)[\overline{Adm}_{h' \xrightarrow{d} X, (\nu b_1, \nu b_2)}],$$

with $h' + \ell(\nu) = h$

As in the previous paragraph, after chasing the normalization sequence for the curve W_0, we obtain, over a connected component of $\mathcal{A}_0$ characterized by covers with ramification profile ν over the shadows of the node, the following decomposition:

$$c_r(\mathcal{I}\mid_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathcal{I}\mid_0'). \tag{16}$$

With this in hand, it is easy to conclude:

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_0}} c_r(\mathcal{I}\mid_0) = \sum_\nu \mathfrak{z}(\nu) \int_{\overline{Adm}_{h' \xrightarrow{d} X, (\nu b_1, \nu b_2)}} c_r(\mathcal{I}\mid_0)$$

$$= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \int_{\overline{Adm}_{h' \xrightarrow{d} X, (\nu b_1, \nu b_2)}} c_{r'}(\mathcal{I}\mid_0')$$

$$= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k_1', k_2')_{\nu,\nu} \quad.$$

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4   Computing the Theory

In order to determine the whole weighted TQFT it is sufficient to compute a small number of invariants, as seen in Fact 3. There are many possible choices for a set of generators; we make the following choice:

generators for the level $(0,0)$ **TQFT:**

1. the coefficients $A_d(0|0,0)_\eta$ of the open (-)disc.

2. the coefficients $A_d(0|0,0)^{\eta,\mu}$ of the (+,+) annulus.

3. the coefficients $A_d(0|0,0)_{\eta,\mu,\nu}$ associated to the (-,-,-) pair of pants.

generators for level shifting :

4. the coefficients of the Calabi-Yau caps $A_d(0|-1,0)_\eta$ and $A_d(0|0,-1)_\eta$.

Theorem 2. *The level* $(0,0)$ *TQFT coincides with the level* $(0,0)$ *theory of Bryan and Pandharipande in [BP04].*

PROOF: It is simple to compute independently the coefficients for the cap. Dimension counts show they are degenerate, in the sense that only the constant term of the series is nonzero. The coefficients for the $(+,+)$-cylinder agree by definition. In section 1 we conclude the proof by showing that the coefficients for the pair of pants are the same.

The significant difference in the theories lies in the Calabi-Yau caps. These are computed by localization on the moduli spaces of admissible covers in section 4.2.

4.1   The Level $(0,0)$ Pair of Pants

The invariants $A_d^\circ(0|0,0)_{\eta,\nu,\mu}$ of the level $(0,0)$ pair of pants are computed by the integrals:

$$\int_{\overline{Adm}^\circ_{h \xrightarrow{d} \mathbb{P}^1, (\eta 0, \mu 1, \nu \infty)}} c^{eq}_{2h-2}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})).$$

The dimension of the moduli space in question is

$$2h - d - 2 + \ell(\eta) + \ell(\mu) + \ell(\nu).$$

Hence, if $\ell(\eta) + \ell(\mu) + \ell(\nu) > d + 2$, the relative connected integrals vanish. The disconnected integrals are then obtained inductively from invariants of lower degree d.

All other invariants have contributions from connected components, and hence need to be computed directly.

In an appendix to [BP04], Bryan, Pandharipande and Faber show that all invariants can be recursively determined from $A_d(0|0,0)_{(d),(d),(2)}$, the invariant

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corresponding to full ramification over two points, and a simple transposition over the third point. Their proof uses only TQFT formalism; hence it suffices to prove the following statement.

Lemma 1. *For $d \geq 2$,*

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{1}{2} \frac{s_1 + s_2}{s_1 s_2} \left(d \cot \left(\frac{du}{2} \right) - \cot \left(\frac{u}{2} \right) \right).$$

Note: The above result differs from the analogous one in [BP04] by a factor of $-i$, that reflects a different normalization in their generating function conventions, that we do not wish to adopt.

PROOF: Notice, first of all, that the full ramification conditions force our covers to be connected. In this case the connected and disconnected invariants coincide.

According to (9), we have:

$$A_d(0|0,0)_{(d),(d),(2)} = \sum_{b_1 + b_2 = 0}^{\infty} u^{b_1 + b_2} s_1^{h-1-b_1} s_2^{h-1-b_2} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty)}} c_{b_1}(\mathbb{E}^*) c_{b_2}(\mathbb{E}^*),$$

with $b_1 + b_2$ equal to the dimension of the moduli space, which is

$$dim(\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty)}) = 2h - 1.$$

For a given value of h, the only nonvanishing terms in the above expression are given by:

- $b_1 = h, \quad b_2 = h - 1$;

- $b_1 = h - 1, \quad b_2 = h$.

Adding the two terms, we obtain

$$A_d^h(0|0,0)_{(d),(d),(2)} = \frac{s_1 + s_2}{s_1 s_2} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty)}} -\lambda_h \lambda_{h-1}$$

and consequently, the generating function:

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{s_1 + s_2}{s_1 s_2} \sum_{h=0}^{\infty} u^{2h-1} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty)}} -\lambda_h \lambda_{h-1} \quad ,$$

where λ_k denotes the kth Chern class of the (pull-back of the) Hodge bundle $\mathbb{E}$.

Let us recall that we defined the λ classes on moduli spaces of admissible covers simply by pulling them back from the appropriate moduli spaces of stable curves. In particular observe the diagram:

$$\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty)} \quad \xrightarrow{\rho} \quad \overline{M}_{h,2}$$

$$\searrow \qquad \downarrow \pi$$

$$\overline{M}_h$$

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The map ρ is defined by marking on the admissible covers the unique preimages of the branch points 0 and 1. The Hodge bundle on $\overline{M}_h$ pulls back to the Hodge bundle on $\overline{M}_{h,2}$, hence we can think of the λ classes on the moduli space of admissible covers as pulled back from $\overline{M}_{h,2}$.

Denote by $H_d \subset M_{h,2}$ the locus of curves admitting a degree d map to $\mathbb{P}^1$ which is totally ramified at the marked points. Let

$$\overline{H}_d \subset \overline{M}_{h,2}$$

be the closure of H_d, consisting of possibly nodal curves admitting a degree d map to a tree of rational curves, fully ramified over the two marked points. The image of the map

$$\rho : \overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0, (d)1, (2)\infty)} \longrightarrow \overline{M}_{h,2}$$

is precisely $\overline{H}_d$, and ρ is a degree $2h$ map onto its image.

From this we conclude that

$$\int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)0, (d)1, (2)\infty)}} -\lambda_h \lambda_{h-1} \quad = 2h \int_{[\overline{H}_d]} -\lambda_h \lambda_{h-1}.$$

This is exactly the integral computed in [BP04], pages 28-29, hence the result follows. This concludes the proof of Lemma 1 and therefore or Theorem 2.

4.2 The Calabi-Yau Cap

First of all let us notice that we can obtain $A_d(0|-1,0)_\eta$ from $A_d(0|0,-1)_\eta$ by simply interchanging the roles of s_1 and s_2.

Theorem 3. *Let d be a positive integer, and $\eta = (\eta_1, \ldots, \eta_{\ell(\eta)})$ a partition of d.*

The degree d Calabi-Yau invariants are :

$$A_d(0|0, -1)_\eta = (-1)^{d - \ell(\eta)} \frac{\left(2 \sin\left(\frac{u}{2}\right)\right)^d}{(s_1)^{\ell(\eta)} \mathfrak{z}(\eta) \prod 2 \sin\left(\frac{\eta_i u}{2}\right)}.$$

Note: In [Cav04], the above formula is computed via localization on moduli spaces of (connected) admissible covers in degree $1, 2, 3$. The result is obtained by finding relations between the Calabi-Yau cap invariants and generating functions for simple Hurwitz numbers. There are two types of obstructions that arise in degree ≥ 4:

1. Fixed loci inside moduli spaces of connected admissible covers are in principle easily described as finite products and quotients of moduli spaces of connected admissible covers, but the combinatorial complexity grows fast.

2. Generating functions for simple Hurwitz numbers are not readily available beyond degree 3.

To circumvent the first problem we interpret the fixed loci in the localization as simpler products of disconnected admissible cover spaces. Then all possible Calabi-Yau invariants, not only the fully ramified ones, appear in the recursions. There is one subtelty to be aware of: Calabi-Yau cap invariants are defined as intersection numbers on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$, whereas the fixed loci are in terms of admissible covers of unparamterized projective lines. Another localization computation, with an appropriate choice of linearizations for the bundles, gives an expression for the invariants in terms of the unparametrized $\mathbb{P}^1$ admissible covers.

To deal with the lack of explicit generating functions for general simple Hurwitz numbers, we notice that the recursive relation that we need to prove is in fact determined by a virtual localization computation on moduli spaces of stable maps. This is yet more evidence of how intimately related this theory and Gromov-Witten theory are.

4.3   Proof of Theorem 3

The Connected Calabi-Yau Cap invariants

We prove the following formula for the connected invariants:

$$
A_d^\circ(0|0, -1)_\eta = \begin{cases} \dfrac{(-1)^{d-1}}{s_1} \dfrac{1}{d} \dfrac{\left(2 \sin\left(\frac{u}{2}\right)\right)^d}{2 \sin\left(\frac{du}{2}\right)} & \text{for} \quad \eta = (d) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{17}
$$

Theorem 3 follows from (17) via exponentiation.

The vanishing of the connected invariants for all partitions but (d) is a dimension count. By definition (8) and formula (9), the genus h contribution to the connected Calabi-Yau invariants is:

$$
A_d^{\circ h}(0|0, -1)_\eta = \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, (\eta\infty)}^\circ} c_{2h+d-1}^{eq}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))
$$

$$
= \sum_{b_1, b_2} s_1^{r_1 - b_1} s_2^{r_2 - b_2} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, (\eta\infty)}^\circ} c_{b_1}(\mathbb{E}^*) c_{b_2}(R^1 \pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) \tag{18}
$$

where:

- $b_1 + b_2 = dim(\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, (\eta\infty)}) = 2h + d + \ell(\eta) - 2;$

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- $r_1 = h - 1$ is the virtual rank of the virtual bundle $-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1})$;

- $r_2 = h+d-1$ is the virtual rank of the virtual bundle $-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))$.

Since
$$-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = R^1 \pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))$$
is in fact a vector bundle of rank $h + d - 1$, we also have the constraint
$$b_1 + b_2 \leq 2h + d - 1.$$

The only possibly nonvanishing integrals occur when $\ell(\eta) = 1$, i.e. when $\eta = (d)$. The indices b_1 and b_2 are forced to be, respectively, h and $h + d - 1$.

Remark: The full ramification condition forces all covers to be connected; the fully ramified connected and disconnected invariants coincide, thus allowing us to drop the superscript "$\circ$".

Finally, our task is to prove:

$$\frac{1}{s_1} \sum_{h=0}^{\infty} u^{2h+d-1} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1, ((d)\infty)}} c_h(\mathbb{E}^*) c_{h+d-1}(R^1 \pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) = \frac{(-1)^{d-1}}{s_1} \frac{1}{d} \frac{\left(2 \sin\left(\frac{u}{2}\right)\right)^d}{2 \sin\left(\frac{du}{2}\right)}.$$

Calabi-Yau Cap Invariants: Parametrized to Unparametrized

We evaluate via localization the Calabi-Yau cap invariant $A_d^h(0|0, -1)_\eta$, for a general partition η.

We linearize the ($\mathbb{C}^*$ action on the) two bundles as indicated in the following table:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	0	1
$\mathcal{O}_{\mathbb{P}^1}$	0	0

There are a priori many fixed loci in the localization computation. However it is possible to rule out a vast majority of them using either dimension counts or linearization considerations (see [Cav04] or [BP05] for a discussion of these standard localization "tricks").

Eventually, the only possibly contributing fixed loci are those whose general element consists of $\ell(\eta)$ spheres S_i, mapping to the main $\mathbb{P}^1$ with degree η_i, all fully ramified over 0 and ∞. A genus 0 twig sprouts from the point ∞ on the main $\mathbb{P}^1$, covered by $\ell(\eta)$ curves C_i of genus h_i. The curve C_i is attached to S_i at a fully ramified point. The h_i's are such that

$$h_1 + \ldots + h_{\ell(\eta)} = h + \ell(\eta) - 1. \tag{19}$$

Finally, if we denote by $F_{\eta,h}$ the disjoint union of all such fixed loci as the h_i's vary, and by N the normal bundle to such fixed loci, we obtain from localization:

$$A_d^h(0|0, -1)_\eta = \int_{F_{\eta,h}} \frac{e^{eq}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) |_{F_{h,\eta}}}{e^{eq}(N)}. \tag{20}$$

Recursion via Localization on Admissible Covers

We now suppose $d > 1$ and consider the following auxiliary integral, computed on the space of connected admissible covers:

$$I^h = \int_{\overline{Adm}^\circ_{h \xrightarrow{d} \mathbb{P}^1}} e^{eq}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))). \tag{21}$$

Elementary dimension reasons give us the vanishing of (21): we are integrating a class whose highest non-equivariant factor has codimension $(2h + d - 1)$ on a space of dimension $2h + 2d - 2$.

On the other hand, if we evaluate the integral via localization we get a relation among Calabi-Yau cap invariants. We let a one dimensional torus act naturally on the moduli space and denote the equivariant parameter s. We choose to linearize the two bundles with weights:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	1	1

The possibly contributing fixed loci E_{η, h_0, h_∞} are represented by connected localization graphs such that any vertex over ∞ has valence 1 ([Cav04]). They can be indexed by triples (η, h_0, h_∞), where

- $\eta = (d_1, \ldots, d_{\ell(\eta)})$ is a partition of d representing the configuration of the spheres over the main $\mathbb{P}^1$;

- h_0 is the genus of the curve lying over 0;

- h_∞ is the genus of the curve lying over ∞ (considered as a disconnected curve);

- $h_0 + h_\infty = h - \ell(\eta) + 1$.

We recognize that a general element in the fixed locus E_{η, h_0, h_∞} is obtained by gluing together an element in the fixed locus F_{η, h_∞} with a connected admissible covers of a genus 0 curve, with a special point of ramification η. Keeping in account the stacky contribution from the gluing, then our integral I on E_{η, h_0, h_∞} reduces to:

$$I^h_{\eta, h_0, h_\infty} = \mathfrak{z}(\eta) \int_{\overline{Adm}^\circ_{h_0 \xrightarrow{d} \mathbb{P}^1, \eta} \times F_{\eta, h_\infty}} \frac{e^{eq}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) \big|_{\left(\overline{Adm}^\circ_{h \xrightarrow{d} \mathbb{P}^1, \eta} \times F_{\eta, h_\infty}\right)}}{e^{eq}(N)} =$$

$$= \mathfrak{z}(\eta) s^{2\ell(\eta)} A^{h_\infty}_d(0|0, -1)_\eta \int_{\overline{Adm}^\circ_{h_0 \xrightarrow{d} \mathbb{P}^1, \eta}} \frac{c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_1) c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_{-1})}{s(s - \psi_\eta)},$$

where $\mathbb{C}_a$ denotes a trivial line bundle where the torus acts on the fibers with weight a.

After expanding the above expression and simplifying using Mumford's relation ([Mum83])

$$c(\mathbb{E})c(\mathbb{E}^*) = 1, \tag{22}$$

we obtain

$$I^h_{\eta,h_0,h_\infty} = \mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A^{h_\infty}_d(0|0,-1)_\eta \int_{\overline{Adm}^\circ_{h_0 \xrightarrow{d} \mathbb{P}^1,\eta}} (-1)^{h_0}\psi^{2h_0+d+\ell(\eta)-4}_\eta =$$

$$= \mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A^{h_\infty}_d(0|0,-1)_\eta \frac{(-1)^{h_0}H^{h_0}_d(\eta)}{(2h_0+d+\ell(\eta)-2)!}. \tag{23}$$

The quantity $H^{h_0}_d(\eta)$ is a simple Hurwitz number, as defined in section 2.4.

The evaluation of the integral I is obtained by adding up the contributions coming from all fixed loci E_{η,h_0,h_∞}:

$$0 = I^h = \sum_{\eta \vdash d} \sum_{h_0+h_\infty=h-\ell(\eta)+1} I^h_{\eta,h_0,h_\infty}. \tag{24}$$

Formula (24) holds for all genera h. All such formulas can be expressed in a very compact form in the language of generating functions. Define:

- $\mathcal{H}_{d,\eta(u)} := \sum \dfrac{(-1)^h H^h_d(\eta)}{(2h+d+\ell(\eta)-2)!} u^{(2h+d+\ell(\eta)-2)}.$

Then formulas (24), for all genera h, are encoded in the relation:

$$0 = \sum_{\eta \vdash d} \mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A_d(0|0,-1)_\eta(u)\mathcal{H}_{d,\eta}(u). \tag{25}$$

Relation (25) determines $A_d(0|0,-1)_{(d)}$ in terms of generating functions for simple Hurwitz numbers and of the invariants $A_d(0|0,-1)_\eta$, for $\ell(\eta) \geq 2$, which can be inductively determined via exponentiation if we assume the theory up to degree $d-1$. The theory has been explicitly computed up to degree 3 in [Cav04], hence the induction can start.

To prove Theorem 3 it therefore suffices to show that relation (25) holds for the conjectured values of the Calabi-Yau invariants. After substituting and simplifying, this amounts to proving:

$$0 = \sum_{\eta \vdash d}(-1)^{\ell(\eta)} \frac{\mathcal{H}_{d,\eta}(u)}{\prod\limits_{\eta_i \in \eta} 2\sin\left(\frac{\eta_i u}{2}\right)}. \tag{26}$$

Virtual Localization on Stable Maps

Relation (26) is the result of explicitly evaluating via virtual localization the following auxiliary integrals:

$$J^h = \int_{\overline{M}_h(\mathbb{P}^1,d)} e^{eq}(-R^\bullet\pi_*f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))). \tag{27}$$

Again dimension reasons grant us the vanishing of this integral. We proceed to linearize the bundles as shown in the following table:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	1	1

The analysis of the possibly contributing fixed loci is parallel to the previous section. The contribution by the fixed locus E_{η,h_0,h_∞} is (see [HKK$^+$03], chapter 27, for a clear and detailed explanation of how to compute these terms, or [BP04], proof of Theorem 5.1 for an extremely similar computation):

$$\sum_{h_1+\ldots+h_{\ell(\eta)}=h_\infty+\ell(\eta)-1} J_{\eta,h_0,h_1,\ldots,h_{\ell(\eta)}},$$

with

$$J_{\eta,h_0,h_1,\ldots,h_{\ell(\eta)}} = \frac{1}{\mathfrak{z}(\eta)} \int_{\overline{M}_{h_0,\ell(\eta)}} \frac{c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_1)c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_1)c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_{-1})}{\prod\left(\frac{1}{\eta_i} - \psi_i\right)}$$

$$\prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h_i,1}} \frac{c_{h_i}(\mathbb{E}^* \otimes \mathbb{C}_1)c_{h_i}(\mathbb{E}^* \otimes \mathbb{C}_1)c_{h_i}(\mathbb{E}^*)}{-\frac{1}{\eta_i} - \psi_1}. \quad (28)$$

After simplifying via Mumford's relation and rearranging things, formula (28) becomes:

$$\frac{(-1)^{h_0}}{Aut(\eta)} \prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h_0,\ell(\eta)}} \frac{1 - \lambda_1 + \ldots \pm \lambda_{h_0}}{\prod(1 - \eta_i\psi_i)} \prod_{i=1}^{\ell(\eta)} -\eta_i^{2h_i-1} \int_{\overline{M}_{h_i,1}} \lambda_{h_i}\psi_1^{2h_i-2}. \quad (29)$$

We recognize in formula (29) two famous results in the field:

ELSV formula: in [ELSV01] and [GV03a], this formula estabilishes the connection between Hurwitz numbers and Hodge integrals:

$$H_d^h(\eta) = \frac{(2h + d + \ell(\eta) - 2)!}{Aut(\eta)} \prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h,\ell(\eta)}} \frac{1 - \lambda_1 + \ldots \pm \lambda_h}{\prod(1 - \eta_i\psi_i)} \quad (30)$$

Faber and Pandharipande's formula: in [FP00], the following class of integrals is computed and expressed in generating function form:

$$\mathcal{L}(u) := \sum u^{2h-1} \int_{\overline{M}_{h,1}} \lambda_h \psi_1^{2h-2} = \frac{1}{2\sin\left(\frac{u}{2}\right)}. \quad (31)$$

Now it is a matter of careful bookkeeping to translate all of the above information in the language of generating functions. Doing so concludes the proof of

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Theorem 3 by estabising the truth of relation (26):

$$0 = \sum_{h \in \mathbb{Z}} J^h u^{2h+2d-2} = \sum_{\eta \vdash d} (-1)^{\ell(\eta)} \mathcal{H}_{d,\eta}(u) \prod_{\eta_i \in \eta} \mathcal{L}(\eta_i u) =$$

$$= \sum_{\eta \vdash d} (-1)^{\ell(\eta)} \frac{\mathcal{H}_{d,\eta}(u)}{\prod_{\eta_i \in \eta} 2 \sin \left(\frac{\eta_i u}{2} \right)} \quad . \tag{32}$$

5   A Specialization of the Theory

We now discuss a specialization of the theory, obtained by embedding a one-dimensional torus inside the two-dimensional torus T, and considering the theory as depending from one equivariant parameter instead of two.

We specialize to the anti-diagonal action, and notice that the coefficients for the product simplify dramatically. It is possible to obtain nice closed formulas for our theory, and to view our TQFT as a one parameter deformation of the classical TQFT of Hurwitz numbers studied by Dijkgraaf and Witten in [DW90] and Freed and Quinn in [FQ93]. Our formulas show connections to the representation theory of the symmetric group S_d. The relevant representation theoretic quantities are introduced in Appendix A.

5.1   The Anti-diagonal Action

Let $\mathbb{C}^*$ be embedded in the two-dimensional torus T via the map

$$\alpha \mapsto \left(\alpha, \frac{1}{\alpha} \right).$$

$\mathbb{C}^*$ acts on N by composing this embedding with the natural action of T constructed in page 13. If we let

$$H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[s],$$

then the one parameter theory obtained with this action corresponds to setting

$$s = s_1 = -s_2.$$

5.1.1   The Q-dimension of an Irreducible Representation

Let ρ be an irreducible representation of the symmetric group on d letters S_d. Classically, a Young diagram, and hence a partition of d, can be canonically associated to ρ (see Appendix A).

We now define the Q-dimension of the representation ρ to be:

$$\frac{dim_Q \rho}{d!} := \prod_{\square \in \rho} \frac{1-Q}{1-Q^{h(\square)}} = \prod_{\square \in \rho} \frac{1}{1+Q+\cdots+Q^{h(\square)-1}} \tag{33}$$

As a consequence of the classical hooklenght formula (36), formula (33) specializes to the ordinary dimension of ρ when setting $Q = 1$.

5.1.2 The Level $(0,0)$ TQFT

The main result is that the level $(0,0)$ TQFT completely collapses to the Dijkgraaf TQFT $\mathcal{D}$. In particular, we have explicit formulas for the semisimple basis of the Frobenius algebra. The basis vectors are indexed by irreducible representations of the symmetric group S_d.

Lemma 2 (essentially Bryan-Pandharipande). *For the anti-diagonal action, the level $(0,0)$ series have no nonzero terms of positive degree in u.*

PROOF: endow $\mathbb{C}$ with the $\mathbb{C}^*$ action

$$\alpha \cdot z = \alpha^n z.$$

This corresponds to considering $\mathbb{C}$ as an equivariant line bundle over a point, whose first equivariant Chern class is ns. We denote such equivariant line bundle by $\mathbb{C}_{ns}$.

The level $(0,0)$ partition functions are, up to some pure weight factor, constructed from integrals of the form:

$$\int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta^1 x_1, \ldots, \eta^r x_r)}} e^{eq}(\mathbb{E}^* \otimes \mathbb{C}_s) e^{eq}(\mathbb{E}^* \otimes \mathbb{C}_{-s}) =$$

$$\int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta^1 x_1, \ldots, \eta^r x_r)}} (-1)^h e^{eq}((\mathbb{E}^* \oplus \mathbb{E}) \otimes \mathbb{C}_s).$$

Equivariant Chern classes of a bundle also are products of ordinary Chern classes times the appropriate factor of s. But by Mumford's relation (22), all Chern classes (but the 0-th) of the bundle $\mathbb{E}^* \oplus \mathbb{E}$ vanish. Hence the only possibly nonvanishing integrals occur when the dimension of the moduli space is 0, which then constitutes the degree 0 term in our generating functions.

We have therefore already essentially produced a semisimple basis for the corresponding Frobenius algebra in page 10. All we need to do is to adjust for the equivariant parameter:

semisimple basis: let ρ be an irreducible representation of the symmetric group S_d, χ_ρ its character function; a semisimple basis for the level $(0,0)$ TQFT is given by the vectors

$$\mathrm{e}_\rho = \frac{dim\rho}{d!} \sum_{\eta \vdash d} (s)^{\ell(\eta) - d} \chi_\rho(\eta) \mathrm{e}_\eta. \tag{34}$$

5.1.3 The Structure of the Weighted TQFT

Theorem 4. *The closed partition functions for the weighted TQFT are given by the following closed formulas:*

$$A_d(g|k_1, k_2) = (-1)^a s^b \sum_\rho \left(\frac{d!}{dim\rho} \right)^{2g-2} \left(\frac{dim\rho}{dim_Q\rho} \right)^{k_1+k_2} Q^{n(\rho)k_1+n(\rho')k_2},$$

$$(35)$$

where:

- *$a = d(g - 1 - k_2)$;*

- *$b = d(2g - 2 - k_1 - k_2)$;*

- *the variable $Q = e^{iu}$;*

- *ρ denotes an irreducible representation of the symmetric group S_d;*

- *ρ' denotes the dual representation;*

- *the function n is defined in section A.1.*

Remark. Notice that by setting $Q = 1$, which corresponds to $u = 0$, we recover the classical formula (6) counting unramified covers of a genus g topological surface. Thus any TQFT naturally embedded in our weighted TQFT constitutes a one parameter deformation of the Dijkgraaf TQFT.

PROOF: by Fact 4, to completely describe the structure of a semisimple weighted TQFT it suffices to evaluate the following quantities:

λ_ρ: the e_ρ-eigenvalue of the genus adding operator, or, equivalently, the inverse of the counit evaluated on e_ρ;

μ_ρ: the e_ρ-eigenvalue of the left level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(0, -1)$ Calabi-Yau cap;

$\overline{\mu}_\rho$: the e_ρ-eigenvalue of the right level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(-1, 0)$ Calabi-Yau cap.

The computation of λ_ρ coincides exactly with Bryan and Pandharipande's

computation in [BP04]. We reproduce it here for the sake of completeness.

$$\lambda_\rho^{-1} = \mathcal{U}\left(\ -\ \begin{matrix} \bigcirc \\ (0,0) \end{matrix}\ \right)(e_\rho)$$

$$= \frac{dim\rho}{d!}\sum_\eta (is)^{\ell(\eta)-d}\chi_\rho(\eta)A_d(0|0,0)_\eta$$

$$= \frac{dim\rho}{d!}(is)^{\ell(1^d)-d}\chi_\rho(1^d)\frac{1}{d!(-s^2)^d}$$

$$= \left(\frac{dim\rho}{d!}\right)^2 (is)^{-2d}$$

Hence,

$$\lambda_\rho = \left(\frac{d!}{dim\rho}\right)^2 (is)^{2d}$$

To compute μ_ρ and $\overline{\mu}_\rho$ let us first of all observe that the tensors associated to the Calabi-Yau caps in our theory are scalar multiples of the tensors in Bryan and Pandharipande's theory.

$$\mathcal{U}(CY\,cap) = 2^d \sin\left(\frac{u}{2}\right)^d \mathcal{BP}(CY\,cap) = \frac{(1-Q)^d}{Q^{\frac{d}{2}}(-i)^d}\mathcal{BP}(CY\,cap).$$

This observation, together with the formulas in [BP04], page 36, implies:

$$\mu_\rho = s^d\frac{d!}{dim\rho}(1-Q)^d s_\rho(Q),$$

$$\overline{\mu}_\rho = (-s)^d\frac{d!}{dim\rho}(1-Q)^d s_{\rho'}(Q),$$

where s_ρ denotes the Schur function of the representation ρ, and is defined to be ([Mac95]):

$$s_\rho(Q) := Q^{n(\rho)}\prod_{\square\in\rho}\frac{1}{1-Q^{h(\square)}}.$$

Plugging this in, we obtain:

$$\mu_\rho = s^d\left(\frac{d!}{dim\rho}\right)(1-Q)^d Q^{n(\rho)}\prod_{\square\in\rho}\frac{1}{1-Q^{h(\square)}}$$

$$= s^d\left(\frac{d!}{dim\rho}\right)Q^{n(\rho)}\prod_{\square\in\rho}\frac{1-Q}{1-Q^{h(\square)}}$$

$$= s^d\left(\frac{dim_Q\rho}{dim\rho}\right)Q^{n(\rho)}$$

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and

$$\overline{\mu}_\rho = (-s)^d \left(\frac{d!}{dim\rho} \right) (1-Q)^d Q^{n(\rho')} \prod_{\square \in \rho'} \frac{1}{1-Q^{h(\square)}}$$

$$= s^d \left(\frac{d!}{dim\rho} \right) Q^{n(\rho')} \prod_{\square \in \rho'} \frac{1-Q}{1-Q^{h(\square)}}$$

$$= s^d \left(\frac{dim_Q\rho}{dim\rho} \right) Q^{n(\rho')}$$

Theorem 4 is finally obtained by using these coefficients in the formula given by Fact 4.

A   Combinatorics and Representation Theory

A.1   Partitions of an Integer

A partition η of an integer d is a finite sequence of positive integers adding up to d:

$$\eta = (\eta^1, \eta^2, \ldots, \eta^r),$$

with

$$\eta^1 \geq \eta^2 \geq \cdots \geq \eta^r$$

and

$$|\eta| = \sum_i \eta^i = d.$$

We use the notation $\eta \vdash d$ to indicate that η is a partition of d.

The number r of nonzero integers is called the *length* of the partition η, and denoted $\ell(\eta)$.

It is also convenient to have a notation that groups all equal parts together. By

$$((\eta^1)^{m_1} \ldots (\eta^k)^{m_k})$$

we denote the partition

$$\eta = (\underbrace{\eta^1, \ldots, \eta^1}_{m_1 \ times}, \underbrace{\eta^2, \ldots, \eta^2}_{m_2 \ times}, \ldots, \underbrace{\eta^k, \ldots, \eta^k}_{m_k \ times}).$$

A partition η of the integer d can be canonically represented by a Young diagram. A *Young diagram* is a left justified array of boxes, such that the lenght of the rows does not increase as you go down the diagram. To a partition η we associate the Young diagram whose i-th row is composed of η^i boxes. For example:

$$(3, 2, 2, 1, 1) = (3^1 2^2 1^2) = \ \ \raisebox{-1em}{\includegraphics{}} \ \ .$$

The conjugate partition η' is the partition associated to the reflection along the main diagonal of the Young diagram associated to η. In our example:

$$(3, 2, 2, 1, 1)' = (5, 3, 1) = \begin{array}{c}\text{[diagram]}\end{array} .$$

Here are some functions associated to Young diagrams that are useful for our purposes:

content: for a given box $\square$ in position (i, j) in a Young diagram, we define the content $c(\square) = i - j$. Boxes on the main diagonal have content 0, boxes above the diagonal have positive content that measures exacly how many diagonals over the main one they lie on, and so on. The *total content $c(\eta)$* is defined to be the sum of the content of all boxes in the diagram. If a diagram is symmetric with respect to the main diagonal, then its total content is 0. In some sense, the total content measures the asymmetry of a Young diagram.

n-**function:** given a Young diagram, we define the n-function as follows: number all boxes in the first row with 0's, all boxes in the second row with 1's, all boxes in the third row with 2's and so on. Then add all of these numbers to obtain $n(\eta)$.

The following obvious formula connects these two quantities:

$$c(\eta) = n(\eta') - n(\eta).$$

hooklength: for a given box $\square$ in a Young diagram, the *hooklength $h(\square)$* is the length (as in number of boxes) of the hook that has the given box as its north-west corner, as shown in Figure 4.
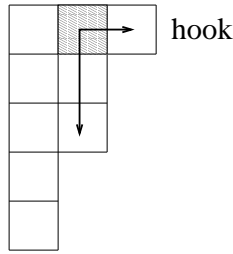

Figure 4: The hook associated to the shaded box.

The total hooklength $h(\eta)$ is the sum of the hooklengths over all boxes in the diagram.

A.2   The Hooklength Formula

Let ρ be an irreducible representation of S_d represented by the Young diagram associated to the partition η. Then the following formula holds:

$$\dim\rho = \frac{d!}{\displaystyle\prod_{\square \in \eta} h(\square)}. \tag{36}$$

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